Functor of points.

Ref: Eisenbud - Harris "Geometry of Schemes" VI

Yoneda embedding: S a scheme (we will take it to be separated over $\mathbb{Z}$, usually an affine)

$$\mathcal{F} : Sch_S \longrightarrow Fun(Sch_S^{op}, Sets)$$

$$X \longmapsto Y \mapsto Hom_S(Y, X) =: X(Y)$$

$\mathcal{F}$ is fully faithful

E.g. rational points: X/k a scheme over a field k (e.g. $\mathbb{Q}$). $X(k)$ (note the importance of considering maps over $\operatorname{Spec} k$!)

E.g. tangent vectors: $\pi^{-1}(x)$ $X(k[\varepsilon])$

$$\begin{array}{c} X(k[\varepsilon]) \\ \downarrow \pi \\ x \in X(k) \end{array} \quad \left(\text{induced by } k[\varepsilon] \to k,\ \varepsilon \mapsto 0\right)$$

tangent vectors at x.

(combined with determining the smoothness of a lfp. scheme/k via its FOP, useful for computing dimension)

Important E.g. Let A be a ring,

$$Hom_A(S, \mathbb{P}^n_A) = \{ \mathcal{O}_S^{\oplus n+1} \twoheadrightarrow \mathcal{L} \}/\sim$$

E.g. Group schemes! (over arbitrary schemes).

for $A = k$, $S = \operatorname{Spec} L$ — normal homogeneous coordinates. description of L-points by

L/k field extension

Linear systems: Ref. Hart. II.7

$\mathcal{L}$ line bundle on ~~scheme~~ X variety over k.

$$V \subseteq \Gamma(X, \mathcal{L})$$

$\dim V = n+1$, V generates $\mathcal{L}$

$$X(k) \ni x \mapsto \ker(V \xrightarrow{f \mapsto \bar{f} \in k(x)} k) \in \mathbb{P}(V^*)(k) \underset{\text{choice of basis of } V}{\cong} \mathbb{P}^n$$

if X is projective $/k$, this is eff. Cartier divisors, smooth the ~~Weil di.~~ eff. Weil divisors —

Elliptic curves over arbitrary field. ~~for maybe check~~

Ref. Hart. IV.4 (see also info in IV.1 and 3)

Weierstrass embeddings:

Let P_0 be the image of e.

$$\begin{array}{c} E \\ \pi \downarrow \uparrow e \\ \operatorname{Spec} k \end{array}$$

$\deg P_0 = 1$.

Consider $\mathcal{O}(P_0) \subseteq \mathcal{O}(2P_0) \subseteq \dots \subseteq \mathcal{O}(nP_0)$

$$\Gamma(E, \mathcal{O}(P_0)) \subseteq --$$

by Riemann-Roch

$\deg K_E = 0$ so $\underset{0}{h_0(-nP_0)} + h_0(nP_0) = \underline{n+1-1 = n}$

pick "1" $\in \Gamma(\mathcal{O}(P_0))$

$x \in \Gamma(\mathcal{O}(2P_0))$

$y \in \Gamma(\mathcal{O}(3P_0))$

~~$z, x, y, xz, \dots$~~

in deg 6 non-zero

$$\underline{y^2, y, \underline{x^3}, x^2, x, 1, yx}$$

$\exists$ so linear relation

so up to change of coordinates $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.

homogenising this gives relation among sections of $\underline{\Gamma(\mathcal{O}(3P_0))}$. very ample. $\underline{\deg 3}$.

Group structure:

~~Assume k alg closed:~~

Ref Hartshorne, Eg. 1.3.7, IV.1
Eg 6.10.2, II.6

$$\begin{array}{ccc} E(\bar{k}) & \xrightarrow{\sim} & Pic_0(E_{\bar{k}}) \\ \uparrow & & \uparrow \pi^* \\ E(k) & \xrightarrow{\sim} & Pic_0(E) \end{array}$$

$3P_0 \sim P+Q+R$

$P \mapsto P - P_0$

or

$P \mapsto P$ addition by $P+Q-P_0$

Check that
$m: E(\bar{k}) \times E(\bar{k}) \longrightarrow E(\bar{k})$
$i: E(\bar{k}) \longrightarrow E(\bar{k})$
come from morphisms.

More natural: S/k consider

$$E \times_k S = E_S \longrightarrow Pic^{(1)}(E_S)$$

fits into the general notion of elliptic curves. Need to understand a bit more...

Flatness

Ref. Hart. III.9

Def^n: $f: X \longrightarrow Y$ morphism of schemes. $\mathcal{F}$ $\mathcal{O}_X$-mod on X. $\mathcal{F}$ is flat over Y at $x \in X$ iff $\mathcal{F}_x$ is flat as a $\mathcal{O}_{Y, f(x)}$-module. Flat iff flat $\forall x \in X$. f flat iff $\mathcal{O}_X$ flat over Y.

Usual good properties: stable under base change, closed under composition:

Relative effective divisors:

Ref Katz-Mazur Ch 1.1-1.2

$$X \to S$$
$\cup$
D $\quad I(D)$ (or $\mathcal{O}(-D)$) invertible, D flat over S.

$$0 \to \mathcal{O}_X \xrightarrow{\ell} I^{-1}(D) \to \underbrace{\mathcal{O}_D \otimes I^{-1}(D)}_{\text{flat}} \to 0$$

($I^{-1}(D)$ or $\mathcal{O}(D)$)

$$\{(\mathcal{L}, \ell)\}/\sim \;\longleftrightarrow\; \text{effective relative Cartier divisor}$$

Flat Pullbacks: $f^* I(D)$ corresponds to base change....
(Also $T \to S$ works).

~~The~~ "Justification": (Cor. 1.1.5.2 in KM, p. 7)

S loc. noeth., X/S flat finite type, D flat$/S$. $\quad D$ effective div
$\iff$
$D \times_S K$ is eff. cartier div in $X \times_S K$ for all geometric points $\operatorname{Spec} K \to X$.

Smoothness

Ref. Néron models, Bosch, Lütkebohmert, Raynaud. Ch. 2

Let $X \xrightarrow{f} S$ be a scheme

Formal smoothness: For any $Y_0 \hookrightarrow Y$ ($\to S$), a closed subscheme of an affine scheme with nilpotent ideal sheaf

~~(i.e. inf. thickening of an affine scheme~~

if $\mathrm{Hom}_S(Y, X) \longrightarrow \mathrm{Hom}_S(Y_0, X)$, we call f formally smooth (bij. form. étale).

Def^n: A loc. fin. pres. map $f: X \longrightarrow S$ is smooth (resp. étale) if it is formally smooth, (resp. étale).

If $S = \mathrm{Spec}\, k$, this corresponds to X being regular for any base change K/k (or for a perfect field extension).

Two other characterisations:

- Flat + smooth fibres (i.e. regular geometric fibres) locally finite presentation (fibres have constant relative dimension. If relative dim $= 0$, then étale)

- X/S, Y/S smooth $f: X \longrightarrow Y$ smooth iff ~~injective~~ surjective on tangent spaces (or injective on cotangent spaces)

$$(f^* \Omega_{Y/S}) \otimes k(a) \twoheadrightarrow \Omega^1_{X/S} \otimes k(x)$$

by relative dimension reasons étale means bijective!

Jacobian criterion: X/S smooth, (rel. dim $= n$.) $j: Y \hookrightarrow X$ closed imm. over S, Y smooth (relative dim r) over S iff

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/S} \otimes \mathcal{O}_Y \to \Omega^1_{Y/S} \to 0 \quad \text{locally split exact}$$

equiv. for all $\cancel{x} = j(y)$, $y \in Y$, $\exists\ g_{r+1}, \dots, g_n$ local ~~eqns for~~ generators of $\mathcal{I}$ at x s.t. $dg_{r+1}(x), \dots, dg_n(x)$ are lin. ind in $\Omega_{X/S} \otimes k(x)$.

Lemma. C/S smooth curve (smooth, proper, ~~geometrically connected~~) any section $s \in C(S)$ defines an effective Cartier divisor.

FP allows to reduce to Noetherian!

Apply ~~previous~~ Cor 1.1.5.2 of K/M.

Back to group structure of elliptic curves: (Ref K/M, Ch 2.1)

E/S elliptic curve. WTS for $T \to S$

$$E(T) \to \mathrm{Pic}^{(1)}(E_T/T) = \left\{ \begin{array}{l} \text{iso classes of} \\ \text{inv. sheaves fibre-wise degree one module} \\ \mathcal{L} \sim \mathcal{L} \otimes_T \mathcal{L}_0 \\ \mathcal{L}_0 \in \mathrm{Pic}(T) \end{array} \right\}$$

given by $P \in E_T(T) = E(T) \longmapsto \mathcal{I}^{-1}(P)$ is an isomorphism!

Conclusion: E/S is a commutative S-group scheme!

Generalized Weierstrass embedding.

Part 1: Riemann-Roch revisited.

- "Cohomological functors"

e.g.
- $H^i(X, \mathcal{F})$
- $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$
- $R^i f_* \mathcal{F}$
- $\underbrace{\mathrm{Ext}^i_f(\mathcal{F}, \mathcal{G})}_{\text{derived functor of } f_* \underline{\mathrm{Hom}}(\mathcal{F}, -)}$.

special case!!! when $X \to \operatorname{Spec} k$. Ref. EGA II, 3.6.

~~Application~~: Grothendieck coherence theorem:

$\mathcal{F}$ coherent, $R^i f_* \mathcal{F}$ is also coherent for $f: X \to Y$ proper.

- idea: use cohomological vanishing for a descending induction
- explicit calculation for $\mathbb{P}^n_A$
  Ref. Hart, III.5
- Chow lemma proj $\Rightarrow$ proper

Chow lemma.
Let $X \to S$ noeth be separated, fin type.
$\exists\ f: X' \to X$ projective ~~[illegible]~~ birational with X' quasi-projective over S.

Cors.
- X/S proper iff ~~[illegible]~~ birationally dominated by proj $/K$
- universal ... only need to check for $\mathbb{P}^n_S$
- proper$/K \Leftrightarrow$ compact

Ref Over Spec k,
Hart. III.7
(H. IV.1 for RR)
– Deligne–Rapoport "Rappels"

Serre duality:

X/k projective, smooth variety. dim n.
$\mathcal{E}$ locally free.

$$H^i(X, \mathcal{E}) \cong H^{n-i}(X, \mathcal{E}^\vee \otimes \omega_X)', \quad \omega_X = \bigwedge^n \Omega_{X/k}$$

Application: Riemann-Roch as $\chi(\mathcal{O}(D)) = \deg D + 1 - g$.

Comes from the following:

X/k proper, a dualizing sheaf on X is a coh. sheaf ω_X° and a trace map $tr: H^n(X, \omega_X^\circ) \to k$ which "represents" $H^n(X, -)'$ in the sense that

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X^\circ) \xrightarrow{tr} k$$

is a perfect pairing.

Then we get natural in $\mathcal{F}$ "duality maps"

$$D^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})'$$

which are isomorphisms if X equidimensional and Cohen–Macaulay (special case: smooth projective variety), when in fact $\omega_X^\circ = \omega_X$.

If $\mathcal{F} = \mathcal{E}$ locally free $\mathrm{Ext}^i(\mathcal{F}, \omega_X) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X)$
$= H^i(X, \mathcal{F}^\vee \otimes \omega_X)$.

Relative Serre duality:

To go from $X \xrightarrow{f} \mathrm{Spec}\, k$
to $X \longrightarrow S$
proper, smooth

Rewrite serre duality as a "derived adjunction"

$$\mathrm{Ext}_f^{\bullet}(\mathcal{F}, \omega_f \otimes N) \to \underline{\mathrm{Hom}}_{\mathrm{Spec}\, k}(R^{n-\bullet} f_* \mathcal{F}, N)$$

In general, it should look like

$f: X \to S$

$\mathcal{N}$ coh on S $\qquad \mathcal{N} \mapsto f^*\mathcal{N} \otimes \omega_f^\bullet$

is "right adjoint" to Rf_*

explicitely natural isos. $\quad D^m : \underline{\mathrm{Ext}}_f^i(\mathcal{F}, \omega_f \otimes f^*\mathcal{N}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_S(R^{m-i}f_*\mathcal{F}, \mathcal{N})$

Applying with $\mathcal{N} = \mathcal{O}_S$ and $\mathcal{F}$ locally free, ... if $R^i f_* \mathcal{F}$ are locally free,

$$(R^{n-i} f_* \mathcal{F})^\vee \cong R^i f_* (\mathcal{F}^\vee \otimes \omega_f)$$

locally free

For relative curves $\omega_f = \Omega^1_{X/S}$, in general for f smooth, proper of relative dim n, $\omega_f = \bigwedge^n \Omega^1_{X/S}$

Cohomology and base change

$f: X \to Y$ separated, quasicompact

$u: Y' \to Y$ flat

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow u \\ X & \xrightarrow{f} & Y \end{array}$$

then $\boxed{u^* R^i f_* \mathcal{F} \cong R^i f'_* (u'^* \mathcal{F})}$ H. III.9

Special case K/k field ext.
$H^i(X, \mathcal{F}) \otimes_k K \cong H^i(X_K, \mathcal{F}_K)$

$$T^i(M) = H^i(X, \mathcal{F} \otimes_A M)$$

more detailed

~~[illegible]~~ For $\mathcal{F}$ flat over Spec A, $M \in Mod_A$

$\downarrow$

X

$$T^i(M) := H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^* \tilde{M})$$

- $T^i(A) \otimes M \longrightarrow T^i(M)$

$\forall M$ iso iff surjective $\forall M$

iff T^i right exact

Ref: Hartshorne, III.12

Cor. T^i ~~[illegible]~~ exact

iff T^i right exact

$+ T^i(A)$ projective

- Using "thm on formal functions" one can check surjectivity just on the fibres!

(there is a nbhd of one fibre result to!)

So we get:

~~Fibres are very far from being flat!~~ geometric points are not flat!

However when $\mathcal{F}$ is flat, coherent, and f is proper, finite pres,

If $\varphi^i(y) : R^i f_*(\mathcal{F}) \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$ surjective then

a) it's an isomorphism

b) $R^i f_*(\mathcal{F})$ locally free in nhood of y $\iff$ $\varphi^{i-1}(y)$ is surjective.

also gives compatibility with all base change

Sketch of proof for Weierstrass embedding. $\boxed{\text{Ref. KM 2.2}}$

$$\begin{tikzcd} E \arrow[d, "f"'] \\ S \arrow[u, bend right, "e"'] \end{tikzcd}$$

~~$f_* \mathcal{O}_E \cong \mathcal{O}_S$ (relative curve property)~~

~~So $R^1 f_* \Omega^1_{E/S} \xrightarrow{\sim} \mathcal{O}_S$~~

locally free

$\underline{\omega_{E/S}}$ $\shortparallel$

~~So $f_* \Omega^1_{E/S}$ is compatible with base change~~

~~hence a line bundle (because $\dim_{k(s)} H^0(E_{k(s)}, \Omega^1_{E_{k(s)}/k(s)}) = 1$)~~

The idea will be to use this to get x, y, etc.. linear relation by fibrewise considerations (where we already know the answers).

Sketch:

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ S \arrow[u, bend left, "e"] \end{tikzcd}$$

$\pi_* \mathcal{O}_E = \mathcal{O}_S$ (follows from connectedness of geometric fibres using "Zariski main theorem")

$R^2 \pi_* \mathcal{O}_E = 0$

$CBC \Rightarrow R^1 \pi_* \mathcal{O}_E$ locally free, compatible with base change!

Grothendieck-Serre duality $\Rightarrow \pi_* \Omega_{E/S}$ locally free!

$\pi^* \pi_* \Omega_{E/S} \longrightarrow \Omega_{E/S}$ is surjective on fibres (easy)

map of ~~free free sheaves~~ invertible sheaves so iso.

- Take ω local basis of $\underline{\omega}_{E/S}$

(so we restrict to $\operatorname{Spec} A \hookrightarrow S$)

formal Completion of E/A at $\operatorname{im} e$ (or "0")

is ~~gives~~ $A[[T]]$ ← formal parameter.

- ~~the~~ ω "adapted to T" iff

$\omega = (1 + O(T))\, dT$. (in $H^0 \hat{\Omega}_{E/A}$).

again loc free comp with base change, etc.

$\pi_* I^n(0) = 0$

$R^2\pi_* I^n(0) = 0$ locally free, compatible with arbitrary base change.

$\Rightarrow R^1\pi_* I^n(0)$ ~~locally free compatible with ...~~

but $R^i\pi_*(I^n(0) \otimes \Omega_{E/S}) = \pi_* \Omega_{E/S} \otimes R^i \pi_* I^n(0)$

so is loc free,

- Serre-Duality $\Rightarrow R^i \pi_* I^{-n}(0)$ are loc. free.

Free in fact!

Because loc free and a "canonical enough" way of picking sections by normalizing "residues" of $S \cdot \omega \in$ $\prod A[[T]][T^{-1}] \otimes \hat{\Omega}_{E/A}$

Fibre by fibre, we then get

arguing

$\pi_* I^{-1}(0)$ $\xrightarrow{\text{free on}}$ "1"

$\pi_* I^{-2}(0)$ $\xrightarrow{\text{free on}}$ $1, x$

$\pi_* I^{-3}(0)$ $\xrightarrow{\text{free on}}$ $1, x, y$

$\pi_* I^{-4}(0)$ free on $1, x, y, x^2$

$\pi_* I^{-5}(0)$ free on $1, x, y, x^2, xy$

two good options for $\pi_* I^{-6}(0)$

$1, x, y, x^2, xy, x^3$ or $\;\cdot\;\;\cdot\;\; y^2$

normalization is such that $x^3 - y^2 \in \pi_* I^{-5}(0)$

so eqn.

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$a_i \in A$.

Check that this is the only eqn (basically fibre by fibre)

Closed imbedding because E/A proper and embedding on fibres.

Q.E.D.