Modular Forms and Schemes Seminar: Talk 1

Evan Marth

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1 Two definitions of modular forms

First, we begin by giving a very general definition of elliptic curve, which one should think of as a family of usual elliptic curves over fields with chosen origins.

Definition 1.1. Let S be a scheme. An ellptic curve E over S is a smooth proper S scheme E with connected geometric fibres of genus 1, together with a section $e : S \to E$.

Let E/S be an elliptic curve with structure morphism π and relative cotangent bundle $\Omega^1_{E/S}$. The sheaf $\underline{\omega}_{E/S} := \pi_*\Omega^1_{E/S}$ is in fact invertible. Since $\pi_*\mathscr{O}_E = \mathscr{O}_S$ for elliptic curves, any non-vanishing differential $\omega_0 \in \Gamma(E, \Omega^1_{E/S})$ gives a trivialization of $\underline{\omega}_{E/S}$. We call such an element a basis for $\underline{\omega}_{E/S}$. Following Katz, we make the following definition:

Definition 1.2. Let R_0 be a ring. A (weakly holomorphic) modular form of level 1 and weight k over R_0 is a rule which assigns to any R_0-scheme S and elliptic curve E over S an element $f(E/S) \in \Gamma(S, \underline{\omega}^{\otimes k}_{E/S})$ such that

i. If $E \cong E'$ are isomorphic as elliptic curves over S, $f(E/S)$ is compatible with $f(E'/S)$ by the induced isomorphism $\underline{\omega}_{E/S} \cong \underline{\omega}_{E'/S}$.

ii. For $\varphi : S' \to S$ a morphism of R_0-schemes, the natural map $\varphi^*\underline{\omega}_{E/S} \to \underline{\omega}_{E\times_S S'/S'}$ defines a functorial pullback map $\varphi^* : \Gamma(S, \underline{\omega}_{E/S}) \to \Gamma(S', \underline{\omega}_{E\times_S S'/S'})$. We require that f be compatible with this in the sense that $\varphi^* f(E/S) = f(E \times_S S'/S')$.

Remark 1.2.1. The rough idea here is that modular forms of weight k should be sections of an invertible sheaf $\underline{\omega}^{\otimes k}$ on the moduli space of elliptic curves over R_0, and each element $f(E/S)$ is the pullback of such a section $\tilde{f}$ coming from the map defined by the elliptic curve E/S. The usual Yoneda philosophy tells us that f should be equivalent data to $\tilde{f}$.

Remark 1.2.2. One can reformulate this definition as follows: first due to the pullback compatibility, only elliptic curves over affine schemes need be considered. Secondly, given a basis ω_0, a section s of $\underline{\omega}^{\otimes k}_{E/\operatorname{Spec} R}$ is determined by the element $s/\omega_0^{\otimes k} \in R$. So a modular form over R_0 of weight k becomes a rule assigning to an R_0-algebra R, and elliptic curve E over $\operatorname{Spec} R$ and a basis ω_0 an element $f(E/R, \omega_0) \in R$ such that this does not depend on a choice of representative from the isomorphism class, is compatible with pullbacks (i.e. for $\varphi : \operatorname{Spec} R' \to \operatorname{Spec} R$, $E' = E \times_{\operatorname{Spec} R} \operatorname{Spec} R'$, $\omega_0' = \varphi^*\omega_0$, $\varphi^\sharp(f(E/R, \omega_0)) = f(E'/R', \omega_0')$), and is homogeneous of degree $-k$ in the second variable (i.e. for $\lambda \in R^\times$, $f(E/R, \lambda\omega_0) = \lambda^{-k} f(E/R, \omega_0)$).

When $R_0 = \mathbb{C}$, this is related to the classical notion of level 1 modular form. Let $H = \{z \in \mathbb{C} : \operatorname{im}(z) > 0\}$ be the complex upper halfplane. It is a homogeneous space for $SL_2(\mathbb{R})$ with the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ given by $z \mapsto \frac{az+b}{cz+d}$.

Definition 1.3. A (weakly holomorphic) modular form of level 1 and weight k over $\mathbb{C}$ is a holomorphic function $f : H \to \mathbb{C}$ such that for $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $f(\mu z) = (cz + d)^k f(z)$.

In both definitions, one can impose a limiting regularity condition ("holomorphic at the cusps"). The subspace of modular forms which obey this condition in both cases are canonically isomorphic.

2 Elliptic curves over $\mathbb{C}$

Over $\operatorname{Spec}\mathbb{C}$, the definition of elliptic curve above reduces to a non-singular, irreducible projective curve E over $\mathbb{C}$ of genus 1 with a chosen (closed) point P_0. As usual, we have the bijection $E(\mathbb{C}) \to \mathrm{Cl}_0(E)$ by $P \mapsto P - P_0$. The induced inversion and multiplication maps on $E(\mathbb{C})$ come from regular morphisms so this makes E into a commutative group variety over $\mathbb{C}$ with identity element P_0. So the analytification of E, E^{an}, is both a connected, compact Riemann surface and a commutative complex Lie group.

Proposition 2.1. *Let E, E' be elliptic curves over $\mathbb{C}$. Then E^{an} (resp. E'^{an}) is isomorphic to T_0E^{an}/Λ, where Λ is a lattice (resp. $T_0E'^{an}$, Λ'). Moreover, morphisms from E to E' sending 0 to 0 (equiv. morphisms $E^{an} \to E'^{an}$ sending 0 to 0, equiv. homomorphisms of complex Lie groups $E^{an} \to E'^{an}$) are precisely those induced by linear maps $T_0E^{an} \to T_0E'^{an}$ sending Λ into Λ'.*

Proof. Use the fact that the exponential map of a connected, commutative Lie group is a surjective homomorphism and that lattices are precisely the discrete cocompact subgroups of a 1-dimensional vector space over $\mathbb{C}$. □

Of course, any quotient $X = \mathbb{C}/\Lambda$ is a compact Riemman surface with $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 2$. It is algebraic (i.e. $X \cong C^{an}$ for an algebraic projective non-singular curve C) and the Hodge decomposition (+ GAGA) implies that $2 = \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{C}} H^0(C, \omega_C) + \dim_{\mathbb{C}} H^1(C, \mathscr{O}_C) = 2g$, so C (+ the obvious origin coming from the iso. $X \cong C^a n$) is an elliptic curve over $\mathbb{C}$. Therefore, to study elliptic curves over $\mathbb{C}$ we can simply look at quotients $\mathbb{C}/\Lambda$ and elements $\alpha \in \mathbb{C}$ which map lattices into one another. In particular, one sees that

Corollary 2.2. *The (analytic) elliptic curves $\mathbb{C}/\Lambda$, $\mathbb{C}/\Lambda'$ are isomorphic iff there exists an $\alpha \in \mathbb{C}^\times$ such that $\alpha\Lambda = \Lambda'$.*

This second condition is easily seen to be an equivalence relation which we call *homothety*. We now define a bijection

$$\{\text{Elliptic curves over } \mathbb{C}\}/\text{iso.} \to SL_2(\mathbb{Z})\backslash H$$

Any lattice Λ in $\mathbb{C}$ is generated by two elements z_1, z_2, and we may always take $z_1/z_2 \in H$. Any other such ordered $\mathbb{Z}$-basis z_1', z_2' is such that there exists $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $z_1' = az_1 + bz_2, z_2' = cz_1 + dz_2$, and conversly if z_1' and z_2' are elements of this form then they are a basis of Λ with $z_1'/z_2' \in H$. Note that $z_1'/z_2' = \mu \cdot (z_1/z_2)$. Thus we get a well-defined map $\{\text{Lattices in } \mathbb{C}\} \to SL_2(\mathbb{Z})\backslash H$ and by construction, two lattices have the same image iff they are homothetic.

2.1 Level structures for $n > 1$

Let $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\tau \in H$. The n-torsion of E_τ are then precisely the points of the form $[\frac{a+b\tau}{n}]$. We have an isomorphism $\alpha_n : (\mathbb{Z}/(n))^2 \to E_\tau[n]$ by $\alpha_n(1, 0) = [1/n]$ and $\alpha_\tau(0, 1) = [\tau/n]$. It is the case that the Weil pairing for E_τ is given by $\langle 1/n, \tau/n\rangle = e^{2\pi i/n}$.

Definition 2.3. A full symplectic level n structure on an elliptic curve E is an isomorphism $\alpha : (\mathbb{Z}/(n))^2 \to E[n]$ such that $\langle \alpha(1, 0), \alpha(0, 1)\rangle = e^{2\pi i/n}$.

Up to isomorphism, all symplectic full level n structures are of the form described above with E_τ and α_τ. Notice that a full (I am going to stop writing symplectic) level n structure (E, α) also gives a canonical element of order n $\alpha(1, 0)$, and such an element gives a order n cyclic subgroup $\langle \alpha(1, 0)\rangle \subseteq E[n]$.

Definition 2.4. Let E be an elliptic curve. A $\Gamma_1(n)$-structure is an element $P \in E[n]$ of order n. A $\Gamma_0(n)$-structure is a cyclic subgroup of $E[n]$ of order n.

Similarly, up to isomorphism all $\Gamma_1(n)$-structures are of the form $(E_\tau, [1/n])$, and all $\Gamma_0(n)$ structures are of the form $(E, \{[m/n] : m = 0, \dots, n-1\})$. These are of course what one gets when restricting level structures from $(E_\tau, \alpha_\tau))$ as described above. Now we may ask when two $\tau, \tau' \in H$ give isomorphic level structures.

Let $n \geqslant 1$. We define the following (finite index!) subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod n \right\}$$

As the notation suggests, these are related to level structures. Consider E_τ and an element $\mu \in \Gamma(n)$. Then by definition, $[\frac{a\tau}{n} + \frac{b}{n}] = [\tau/n] \in E[n]$ and $[\frac{c\tau}{n} + \frac{d}{n}] = [1/n] \in E[n]$. In other words, the isomorphism $E_{\mu\tau} \to E_\tau$ given by $(c\tau + d) \in \mathbb{C}$ is compatible with the level structures α_τ and $\alpha_{\mu\tau}$. Similarly, given such an isomorphism between an E_τ and an $E_{\tau'}$, by lifting to $\mathbb{C}$ and looking at the images of τ/n and $1/n$, written in the basis $\{\tau'/n, 1/n\}$, one gets an element of $\mu \in \Gamma(n)$ with $\mu\tau = \tau'$. So the isomorphism classes of full level n structures are given by $\Gamma(n)\backslash H$. In a similar fashion, $\Gamma_1(n)$ and $\Gamma_0(n)$ orbits of H give the isomorphism classes of $\Gamma_1(n)$ and $\Gamma_0(n)$ structures (you can "read" what level structure is being preserved off the congruence). The situation can be summarized as follows, with the vertical arrows being the obvious quotient maps and the restrictions of level structure discussed above:

$$\begin{tikzcd}
\Gamma(n)\backslash H \arrow[r, "\tau\mapsto(E_\tau,\alpha_\tau)"'] \arrow[d] & \{\text{Elliptic curves over } \mathbb{C} \text{ with full level } n \text{ structure}\}/\sim \arrow[d] \\
\Gamma_1(n)\backslash H \arrow[r, "\tau\mapsto(E_\tau,[1/n])"'] \arrow[d] & \{\text{Elliptic curves over } \mathbb{C} \text{ with } \Gamma_1(n)\text{-structure}\}/\sim \arrow[d] \\
\Gamma_0(n)\backslash H \arrow[r, "\tau\mapsto(E_\tau,\langle[1/n]\rangle)"'] \arrow[d] & \{\text{Elliptic curves over } \mathbb{C} \text{ with } \Gamma_0(n)\text{-structure}\}/\sim \arrow[d] \\
SL_2(\mathbb{Z})\backslash H \arrow[r, "\tau\mapsto E_\tau"'] & \{\text{Elliptic curves over } \mathbb{C}\}/\sim
\end{tikzcd}$$

3 Modular curves

Definition 3.1. A subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgroup if it contains $\Gamma(n)$ for some $n \geqslant 1$. $\overline{\Gamma}$ is the image of Γ in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$. $\mu_\Gamma = [PSL_2(\mathbb{Z}) : \overline{\Gamma}]$.

Note that the $\Gamma_0(n), \Gamma_1(n)$ are congruence subgroups. We want to endow $\Gamma\backslash H$ with the structure of a complex manifold. First, we look at the action of $G = PSL_2(\mathbb{Z})$ on H. G is generated by two elements:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Proposition 3.2. *Let* $D = \{\tau \in H : |\mathrm{Re}(\tau)| \leqslant 1/2, |\tau| \geqslant 1\}$. *Every element of* H *is* G*-conjugate to an element in* D. *Two distinct elements* $\tau, \tau' \in D$ *are conjugate iff* $|\mathrm{Re}(\tau)| = |\mathrm{Re}(\tau')| = 1/2$ *(in which case* $\tau = \tau' \pm 1$*) or* $|\tau| = |\tau'| = 1$ *(in which case* $\tau = \frac{-1}{\tau'}$*). Moreover, all elements of* D *except* $i, \rho = e^{2\pi i/3}$ *and* $\rho' = \rho + 1 = e^{2\pi i/6}$ *have trivial stabalizer, with* $\mathrm{Stab}(i) = \langle S \rangle$, $\mathrm{Stab}(\rho) = \langle ST \rangle$, *and* $\mathrm{Stab}(\rho') = \langle TS \rangle$.

In particular, the quotient map $\pi : H \to SL_2(\mathbb{Z})\backslash H =: Y(1)$ is a covering space away from $\pi(i)$ and $\pi(\rho)$. Thus, for any neighbourood $U \subseteq Y(1)$ which does not contain these points, we can take the holomorphic functions on U to be exactly the functions which pullback to holomorphic functions on $\pi^{-1}(U)$ to define a complex structure. This can be extended to all of $Y(1)$ in such a way that π is holomorphic and locally at any $z_0 \in \{i, \rho, \rho'\}$, there are neighbourhoods U of z_0, V of $\pi(z_0)$ and isomorphisms $\varphi : B \to U, \psi : B \to V$ with $B = \{z \in \mathbb{C} : |z| < 1\}$, $\varphi(0) = z_0$, $\psi(0) = \pi(z_0)$ such that

$$\begin{tikzcd} B \arrow[r, "z \mapsto z^d"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ U \arrow[r, "\pi|_U"'] & V \end{tikzcd}$$

commutes for some $d \in \mathbb{N}$. In particular, $d = 2$ when $z_0 = i$ and $d = 3$ otherwise.

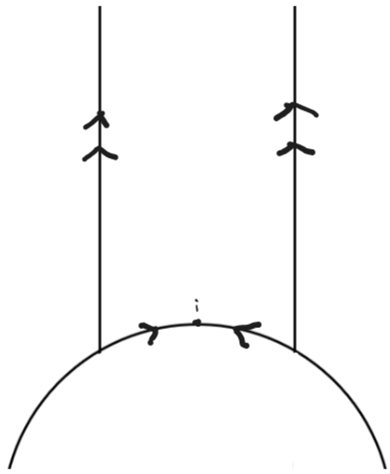
A similarly, for the quotient map $\pi : H \to \Gamma\backslash H =: Y_\Gamma$, it is unramified away from the points in H which have non-trivial stabilizer in $\bar{\Gamma}$, and Y_Γ can be given a complex structure such that the projection is holomorphic, and locally at ramifications admits the same description.

The quotients Y_Γ admit compactifications X_Γ defined by $\Gamma\backslash H^*$, where $H^* = H \cup \mathbb{P}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{C})$. X_Γ admits a structure of a compact Riemann surface for which Y_Γ with the previously defined complex structure is an open complex submanifold. The points in the image of $\mathbb{P}(\mathbb{Q})$ in X_Γ are called the cusps of X_Γ.

Proposition 3.3. *For $\Gamma = SL_2(\mathbb{Z})$, $X_\Gamma = X(1)$ has a unique cusp and is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.*

Proof. For the first claim, it is enough to show that for any $r \in \mathbb{Q}$, there exists $A \in SL_2(\mathbb{Z})$ such that $A \cdot \infty = r \in \mathbb{P}^1(\mathbb{Q})$. Write $r = m/n$ with m, n coprime. Then there are $a, b \in \mathbb{Z}$ such that $an - bm = 1$, and so $\begin{pmatrix} a & m \\ b & n \end{pmatrix}$ gives the desired A.

Now, we have that $D \cup \{\infty\}$ (D the fundamental domain from Proposition 3.2) maps surjectively onto $X(1)$, with the following identifications:



Taking any z from the interior of D and connecting it to ρ, i and ∞ then shows that $X(1)$ is homeomorphic to a tetrahedron, so in particular has genus 0 and so must be $\mathbb{P}^1_{\mathbb{C}}$. □

With this information, we can compute the genus of X_Γ by applying the Hurwitz formula to the quotient map $\pi_\Gamma : X_\Gamma \to X(1)$. This works out to

Theorem 3.4. *Let ν_2 for the number of points in the fibre $\pi_\Gamma^{-1}([i])$ for which π_Γ is unramified, ν_3 for the number of points in the fibre $\pi_\Gamma^{-1}([\rho])$ for which π_Γ is unramified, and ν_∞ for the number of cusps of X_Γ, i.e. the cardinality of $\pi_\Gamma^{-1}([\infty])$. The genus of X_Γ is given by*

$$g_\Gamma = 1 + \frac{\mu_\Gamma}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}$$

4 Modular forms

Delayed to next talk.

5 Application: sum of four squares

Delayed to next talk.