Irrational Rotation Algebras

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Abstract

Let θ be an irrational number. Rotation on the circle S^1 by an angle of $2\pi\theta$ gives a minimal dynamical system. To this, one can associate a C^*-algebra A_θ called the *irrational rotation algebra*. In this note, we shall show it is a complete invariant of the dynamical system by computing the image of the unique induced trace map on $K_0(A_\theta)$.

Conventions: *All C^*-algebras are unital and all $*$-homomorphisms send the 1 to 1.*

1 Irrational Rotations and their Algebra

We begin by observing that it makes no difference to simply work with an irrational number $\theta \in [0,1]$ acting by addition on $\mathbb{R}/\mathbb{Z}$ due to the isomorphism of topological groups $t \mapsto e^{2\pi i t}$. With this in mind, we make the following definition.

Definition 1.1. *Let $X = \mathbb{R}/\mathbb{Z}$ and $\theta \in [0,1]$. We write (X, θ) for the dynamical system on X given by $\varphi : X \to X$, $\varphi(t) = t + \theta$.*

The dynamical system (X, θ) will thus correspond to rotation by an angle of $2\pi\theta$ on $S^1 \subseteq \mathrm{GL}(1, \mathbb{C})$.

We now turn to the question of which choices of θ give equivalent (that is to say, topologically conjugate) dynamical systems. Clearly, on the circle the choice to rotate clockwise or counter-clockwise should not matter. More precisely,

Proposition 1.2. *Let $\theta \in [0,1]$ be an irrational number. Then (X, θ) is topologically conjugate to $(X, 1-\theta)$.*

Proof. Let φ and φ' be the rotations from (X, θ) and $(X, 1-\theta)$ respectively, and let $\psi : X \to X$, with $\psi(t) = -t$. The latter is well-defined and an involution. So we must show that $\psi \circ \varphi \circ \psi = \varphi'$, i.e. for all $t \in X$, $-(-t + \theta) = t + 1 - \theta$. But this is true since the representatives differ by 1. $\square$

Our main result is that this is the only case.

Theorem 1.3. *Let $\theta, \theta' \in [0,1]$ be irrational numbers. Then the following are equivalent:*

(i) (X, θ) *is topologically conjugate to* (X, θ').
(ii) $\theta = 1 - \theta'$
(iii) $\mathbb{Z} + \mathbb{Z}\theta = \mathbb{Z} + \mathbb{Z}\theta'$

The equivalence of (ii) and (iii) is clear and we have just seen (ii) implies (i). Our goal will be to show (i) implies (iii). We begin by defining the key invariant.

Definition 1.4. *Let (X, θ) be an irrational rotation and let $\varphi : X \to X$ be the rotation map. Then there is an induced operator $V := (\varphi^*)^{-1} : H \to H$, where $H := L^2(X)$ is the C^*-algebra of square integrable functions on X with respect to the uniform Haar measure. Let $M : C(X, \mathbb{C}) \to B(H)$ be the $*$-homomorphism sending a continuous function f to the operator $M_f \in B(H)$ defined by $g \mapsto fg$. We define the (concrete) irrational rotation algebra A_θ as $C^*(M(C(X, \mathbb{C})), V)$.*

Theorem 1.5. *If (X, θ) is topologically conjugate to (X, θ'), then $A_\theta \cong A_{\theta'}$.*

To prove this theorem, we must first establish some properties of A_θ.

By the complex version of the Stone-Weierstrass theorem, we have that $C(X, \mathbb{C}) = C^*(z)$, where $z : X \to \mathbb{C}$ is $t \mapsto e^{2\pi i t}$. Since M is a $*$-homomorphism, we have that $A_\theta = C^*(M_z, V)$. Notice that $z\overline{z} = \overline{z}z = 1$, so $U := M_z$ is unitary. The elements z^n (that is the classes in $L^2(X)$ of $t \mapsto e^{2\pi n i t}$) for $n \in \mathbb{Z}$ are an orthonormal Hilbert space basis for H, so the fact that $V(z^n) = e^{-2\pi n i \theta} z^n$ implies that V is unitary. We have the following important relation:

$$UV = e^{2\pi i \theta} VU \tag{1.1}$$

This motivates the following definition.

Definition 1.6. *An* abstract irrational rotation algebra *for a given irrational $\theta \in [0, 1]$ is a triple (A, U, V), with C^*-algebra A, U, V unitary elements of A which generate it and satisfy (1.1).*

Proposition 1.7. (Universal abstract irrational rotation algebra): *There exists an abstract irrational rotation algebra $\left(\mathcal{A}_\theta, \tilde{U}, \tilde{V}\right)$ for $\theta \in [0, 1]$ such that for any other abstract irrational rotation algebra (A, U, V), there is a $*$-homomorphism $f : \mathcal{A}_\theta \to A$ with $f(\tilde{U}) = U$ and $f(\tilde{V}) = V$.*

Proof. Consider the set[1] I of pairs of unitary operators (U, V) on a Hilbert space satisfying (1.1). So to every element $\alpha \in I$, we associate H_α a Hilbert space and U_α, V_α the unitary operators. Consider $\mathcal{H} := \bigoplus_{\alpha \in I} H_\alpha$ and the induced unitary operators $\tilde{U}, \tilde{V}$ on H. We define $\mathcal{A}_\theta = C^*\left(\tilde{U}, \tilde{V}\right) \subseteq B(\mathcal{H})$. Now let (A, U, V) be an abstract irrational rotation algebra. There exists a faithful $*$-representation π of A on a Hilbert space. Together with the images $\pi(U), \pi(V)$, this is one of the elements of I. Thus given a non-commutative polynomial $p \in T(\mathbb{C}^4)$, $\|p(U, V, U^*, V^*)\| = \|p(\pi(U), \pi(V)), \pi(U)^*, \pi(V)^*)\| \leq \|p(\tilde{U}, \tilde{V}, \tilde{U}^*, \tilde{V}^*)\|$ by the faithfulness of π and the construction of $\tilde{U}, \tilde{V}$. The following lemma completes the proof. □

Lemma 1.8. *Let A, B be C^*-algebras, A generated by elements $x_1, ..., x_n$, and $y_1, ..., y_n \in B$ such that for all non-commutative polynomials $p \in T(\mathbb{C}^{2n})$ $\|p(y_1, ..., y_n, y_1^*, ..., y_n^*)\| \leq \|p(x_1, ..., x_n, x_1^*, ..., x_n^*)\|$, then there is a well-defined $*$-homomorphism $f : A \to B$ with $f(x_i) = y_i$ $(i = 1, ..., n)$.*

Proof. Let $A' \subseteq A$ be the $*$-subalgebra generated by $x_1, ..., x_n$. Then for any $a \in A'$, there is a polynomial $p \in T(\mathbb{C}^{2n})$ such that $a = p(x_1, ..., x_n, x_1^*, ..., x_n^*)$. We define $f'(a) = p(y_1, ..., y_n, y_1^*, ..., y_n^*)$. This is well-defined since if there were another polynomial p' with which to represent a, then

$$\|p(y_1, ..., y_n, y_1^*, .., y_n^*) - p'(y_1, ..., y_n, y_1^*, .., y_n^*)\| \leq \|p(x_1, ..., x_n, x_1^*, .., x_n^*) - p'(x_1, ..., x_n, x_1^*, .., x_n^*)\| = 0$$

since $p - p'$ is also a polynomial, hence f' is a well-defined $*$-homomorphism $A' \to B$. Notice that the inequality also shows that f' maps Cauchy sequences to Cauchy sequences. By our

[1] This does not actually form a set, however the pairs needed for the proof to work do.

hypothesis on $x_1, ..., x_n$, $A' \hookrightarrow A$ is the Cauchy completion of A', so f' extends uniquely to a $*$-homomorphism $f : A \to B$. Clearly $f(x_i) = y_i$. $\square$

***Remark* 1.9.** The $*$-homomorphism of Proposition 1.7 is also unique since $\tilde{U}$ and $\tilde{V}$ generate $\mathcal{A}_\theta$. This implies in turn that all universal irrational rotations algebras (for fixed θ) are canonically isomorphic.

We now study the[2] universal irrational rotation algebra $\mathcal{A}_\theta$, starting with a useful family of automorphisms.

Let $\lambda, \mu \in \mathbb{C}$ with $|\lambda| = |\mu| = 1$. Then $\lambda\tilde{U}, \mu\tilde{V}$ are both unitary and still satisfy (1.1). Define $\rho_{\lambda,\mu} : \mathcal{A}_\theta \to \mathcal{A}_\theta$ as the unique $*$-homomorphism with $\rho_{\lambda,\mu}(\tilde{U}) = \lambda\tilde{U}$ and $\rho_{\lambda,\mu}(\tilde{V}) = \lambda\tilde{V}$. Notice that $\rho_{\lambda,\mu} \circ \rho_{\overline{\lambda},\overline{\mu}} = \rho_{\overline{\lambda},\overline{\mu}} \circ \rho_{\lambda,\mu} = \mathrm{Id}$, so they are automorphisms of $\mathcal{A}_\theta$.

Lemma 1.10. *Fix $X \in \mathcal{A}_\theta$. Then the map $f : \mathbb{T}^2 \to \mathcal{A}_\theta$, $(\lambda, \mu) \mapsto \rho_{\lambda,\mu}(X)$ is continuous.*

Proof. This is clear for X which are non-commutative polynomials in $\tilde{U}^{\pm 1}$ and $\tilde{V}^{\pm 1}$ (reduces to closure of continuous functions under algebraic operations). For an arbitrary $X \in \mathcal{A}_\theta$, one can find a sequence $\left(X_i = p_i(\tilde{U}, \tilde{V}, \tilde{U}^*, \tilde{V}^*)\right)_{i\in\mathbb{N}}$, p_i non-commutative polynomials, which converges to X. Since the $\rho_{\lambda,\mu}$ are $*$-automorphisms, they are isometric so the supremum norm $\sup_{\lambda,\mu\in\mathbb{T}} \|\rho_{\lambda,\mu}(A) - \rho_{\lambda,\mu}(B)\|$ is just $\|A - B\|$ for any $A, B \in \mathcal{A}_\theta$, hence the continuous functions given by the X_i converge uniformly to the continuous function given by X, which must then be continuous. $\square$

Now we define the following two operators on $\mathcal{A}_\theta$:

$$\Phi_1(X) = \int_0^1 \rho_{1,e^{2\pi it}}(X)dt \qquad \Phi_2(X) = \int_0^1 \rho_{e^{2\pi it},1}(X)dt$$

where this is just the Riemann integral of a continuous Banach space valued function on $[0, 1]$.

Proposition 1.11. *Φ_1 is a contractive, positive, idempotent and faithful linear map with image $C^*(\tilde{U})$. For any $n, m \in \mathbb{Z}$, $\Phi_1(\tilde{U}^n\tilde{V}^m) = \delta_{0m}\tilde{U}^n$. For any $X \in \mathcal{A}_\theta$, $\Phi_1(X) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \tilde{U}^j A \tilde{U}^{-j}$. For Φ_2, the same results hold with $\tilde{V}$ switched with $\tilde{U}$*

Proof. Linearity is clear. It is contractive since for any $X \in \mathcal{A}_\theta$, $\| \int_0^1 \rho_{1,2\pi it}(X)dt\| \leq \int_0^1 \|\rho_{1,2\pi it}(X)\| \; dt \leq \|X\|$ since the $\rho_{\lambda,\mu}$ are isometric. If X is positive, then so is $\rho_{1,2\pi it}$ for all $t \in [0, 1]$, so all the Riemann sums approximating $\Phi_1(X)$ are positive. Since the set of positive elements in a C^*-algebra is closed, $\Phi_1(X)$ is also positive. If X is also non-zero, $\rho_{1,2\pi it}(X)$ will be as well for all t, and since these are positive the integral will be non-zero[3], hence Φ_1 is faithful.

Given $n, m \in \mathbb{Z}$, one simply computes

$$\Phi_1(\tilde{U}^n\tilde{V}^m) = \int_0^1 \rho_{1,e^{2\pi it}}(\tilde{U}^n\tilde{V}^m)dt = \left(\int_0^1 e^{2\pi imt}dt\right)\tilde{U}^n\tilde{V}^m$$

The integral is δ_{0m} from which the desired equality follows. From this, one sees the image of Φ_1 is in $C^*(U)$ and that it is idempotent.

[2]This makes sense given *Remark* 1.9

[3]According to the proof of [1, Theorem VI.1.1, p. 168]. I could not verify this myself.

For the last formula, first notice that function which assigns to X the expression on the RHS is continuous and linear, since it is a uniform limit of the continuous linear functions. Since Φ_1 is also linear and continuous, it is enough to prove the equality for a monomial $\tilde{U}^n\tilde{V}^m$, $n, m \in \mathbb{Z}$. In this case we have

$$\frac{1}{2n+1}\sum_{j=-n}^{n}\tilde{U}^j\big(\tilde{U}^n\tilde{V}^m\big)\tilde{U}^{-j} = \frac{1}{2n+1}\sum_{j=-n}^{n}e^{2\pi ijm\theta}\tilde{U}^n\tilde{V}^m$$

by applying (1.1) jm times to move j $\tilde{U}$s to the right of the m $\tilde{V}$s. It is enough now to show that $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}e^{2\pi ijm\theta} = \delta_{0m}$ The equality is clear for $m = 0$. Suppose $m \neq 0$, then since θ is irrational, $e^{2\pi im\theta} \neq 1$. We have that

$$e^{2\pi im\theta}\sum_{j=-n}^{n}e^{2\pi ijm\theta} = \sum_{j=-n}^{n}e^{2\pi ijm\theta} + e^{2\pi i(n+1)m\theta} - e^{-2\pi inm\theta}$$

whence

$$\frac{e^{2\pi im\theta}-1}{2n+1}\sum_{j=-n}^{n}e^{2\pi ijm\theta} = \frac{e^{2\pi i(n+1)m\theta} - e^{-2\pi inm\theta}}{2n+1}$$

The limit of the norm of the RHS as $n \to \infty$ is clearly 0, and since $e^{2\pi im\theta} - 1 \neq 0$, we thus have $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}e^{2\pi ijm\theta} = 0$. The proofs are essentially the same for Φ_2 and $\tilde{V}$. $\square$

Theorem 1.12. *The universal irrational rotation algebra $\mathcal{A}_\theta$ is simple.*

Proof. Suppose $I \subseteq \mathcal{A}_\theta$ is a non-zero ideal. Then there is a positive element $X \in I$. Then $\Phi_1(X) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}\tilde{U}^jx\tilde{U}^{-j}$ is also in I, and by the same argument $\Phi_2(\Phi_1(X)) \in I$. However, since $\Phi_2 \circ \Phi_1$ maps monomials of $\tilde{U}, \tilde{V}$ into $\mathbb{C}\cdot 1$ and are both positive and faithful, we have that $\Phi_2(\Phi_1)(X) = a\cdot 1$, $a \neq 0$, so $I = \mathcal{A}_\theta$. $\square$

Corollary 1.13. *All non-zero abstract irrational rotation algebras for a fixed θ are canonically isomorphic. In particular, the concrete irrational rotation algebra A_θ is universal.*

Due to the canonical isomorphism, we may use freely the results and functionals on $\big(\mathcal{A}_\theta, \tilde{U}, \tilde{V}\big)$ with (A_θ, U, V).

Proof of Theorem 1.5. Let φ and φ' denote the rotation maps for (X, θ) and (X, θ') repsectively, and suppose a homeomorphism $\psi : X \to X$ gives a topological conjugacy, i.e. $\psi\circ\varphi = \varphi'\circ\psi$, which gives $\mathbb{Z}$-equivariance. Applying this for -1 gives $\psi(t-\theta) = \psi(t) - \theta'$. $U' := M_{z\circ\psi}$ and V are unitaries which generate A_θ, since ψ is a homeomorphism. For any $f \in L^2(X)$ we have

$$(U'V)(f) = e^{2\pi i\psi(t)}f(t-\theta), \qquad (VU')(f) = e^{2\pi i\psi(t-\theta)}f(t-\theta) = e^{-2\pi i\theta'}\cdot e^{2\pi i\psi(t)}f(t-\theta)$$

So U', V satisfy (1.1) for θ', hence A_θ is a non-zero abstract irrational rotation algebra for θ', so $A_\theta \cong A_{\theta'}$ $\square$

2 Traces and K_0

First, we recall the positive homomorphism from the K_0 group of an algebra to the real numbers induced by a trace.

Definition 2.1. *Let A be a C^*-algebra and $\tau : A \to \mathbb{C}$ a linear map. We say that τ is a trace on A iff τ is bounded, positive and satisfies $\tau(xy) = \tau(yx)$ for any $x, y \in A$. It is a tracial state if in addition $\tau(1_A) = 1$.*

***Example* 2.2.** The usual trace on the matrix algebra $M_n(\mathbb{C})$ is a trace in the sense of Definition 2.1. Scaling by $\frac{1}{n}$ yields a tracial state (in fact, the only one for $M_n(\mathbb{C})$). More generally, given A a C^*-algebra and τ a trace on A, the map $\tau^{(n)} : M_n(A) \to \mathbb{C}$, $\tau^{(n)}\big((x_{ij})\big) = \sum_{i=1}^n \tau(x_{ii})$ is a trace.

From the defining properties, we get the following result (see also the discussion in [2, p. 46]).

Proposition 2.3. *Let τ be a trace on a C^*-algebra A. The $\tau^{(n)}, n \geq 0$ from Example 2.2 fit together to define a map $\tau^{(\infty)} : M_\infty(A) \to \mathbb{C}$. The restriction of this map to projections in M_∞ has image in the positive real numbers, sends 0 to 0, sends direct sums to sums, and if P, Q are Murray-von Neumann equivalent projectors, $\tau^{(\infty)}(P) = \tau^{(\infty)}(Q)$. In particular, there is an induced positive homomorphism $\tau_* : (K_0(A), K_0(A)^+) \to (\mathbb{R}, \mathbb{R}^+)$. If τ is a tracial state, then τ_* is a state with respect to the order unit $[1_A]$.*

Proposition 2.4. *There is a unique tracial state on A_θ.*

Proof. Existence: From Proposition 1.11, we see that $\Phi_1 \circ \Phi_2$ sends the $*$-subalgebra of A_θ generated by U, V to $\mathbb{C} \cdot 1$. Since this space is closed, it is also the image of $\Phi_1 \circ \Phi_2$, and using the identification $\mathbb{C} \ni a \mapsto a \cdot 1 \in A_\theta$, we get a linear, positive bounded functional, which we write τ. By continuity and linearity, it suffices to verify the relation $\tau(xy) = \tau(yx)$ with x, y monomials in U, V. We have $\tau\big((U^k V^l)(U^m V^n)\big) = e^{-2\pi i l m \theta} \tau\big(U^{k+m} V^{l+n}\big)$ and $\tau\big((U^m V^n)(U^k V^l)\big) = e^{-2\pi i n k \theta} \tau\big(U^{k+m} V^{l+n}\big)$ Both are zero unless $k + m = l + n = 0$, so equality is automatic in this case. If $k + m = l + n = 0$, then we have $kn = lm$, so again the equality holds. It is a tracial state since $\tau(1) = \tau(1 \cdot U^0 V^0) = 1$.

Uniqueness: Let $\sigma : A_\theta \to \mathbb{C}$ be a tracial state. Then by linearity, continuity (it is bounded) and invariance under conjugation (follows from $\sigma(xy) = \sigma(yx)$) we must have for all $X \in A_\theta$

$$\sigma(X) = \lim_{n \to \infty} \sigma\left(\frac{1}{2n+1} \sum_{j=-n}^{n} U^j A U^{-j}\right) = \sigma(\Phi_1(A))$$

and by the same argument, $\sigma(X)) = \sigma(\Phi_2(X))$. Therefore

$$\sigma(X) = \sigma(\Phi_2(X)) = \sigma(\Phi_1(\Phi_2(X))) = \sigma(\tau(X) \cdot 1) = \tau(X)\sigma(1) = \tau(X)$$

$\square$

The uniqueness implies that the image of τ_* in $\mathbb{R}$ is an invariant of A_θ. To prove $(i) \Rightarrow (iii)$ from Theorem 1.3, we will show that this image is $\mathbb{Z} + \mathbb{Z}\theta$. The first step towards this is the following theorem of Rieffel:

Theorem 2.5. *For every $\alpha(0, 1)$ of the form $[\alpha] = [k\theta]$ in $\mathbb{R}/\mathbb{Z}$, there is a projection $P \in A_\theta$ with $\tau(P) = \alpha$.*

Proof. Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. For a real number a, write f_a for $f(t - a)$. $M_f \in C^*(U) \subset A_\theta$ can be written out as a Fourier series, and applying τ will give the zeroeth Fourier coefficient by Proposition 1.11, i.e. $\tau\big(M_f\big) = \int_0^1 f(t)dt$. For an element of the form $M_g V + M_f + M_h V^*$, $f, g, h \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ the trace would be the same. We use this form as an ansatz for a projection. Self-adjointness means that

$$M_g V + M_f + M_h V^* = V^* M_{\overline{g}} + M_{\overline{f}} + V M_{\overline{h}} = M_{\overline{g}_{-\theta}} V^* + M_{\overline{f}} + M_{\overline{h}_\theta} V$$

gives necessary and sufficient functional equations $f = \overline{f}$, i.e. f is real-valued, and $g(t) = \overline{h(t-\theta)}$. By a similar argument (which is better left unwritten) idempotence (and the above relation of g and h) gives necessary and sufficient the functional equations

$$g(t)g(t-\theta) = 0 \tag{2.1}$$

$$g(t)(1-f(t)) - f(t-\theta) = 0 \tag{2.2}$$

$$f(t) - f(t)^2 = |g(t)|^2 + |g(t+\theta)|^2 \tag{2.3}$$

To begin with we solve this system in the case $0 < \theta < \frac{1}{2}$ with a judicious choice of f. Since $\theta < \frac{1}{2}$, one can fix an $\varepsilon > 0$ such that $\theta + \varepsilon < \frac{1}{2}$ and $\varepsilon < \theta$. Let

$$f(t) := \begin{cases} \frac{t}{\varepsilon} & 0 \le t \le \varepsilon \\ 1 & \varepsilon \le t \le \theta \\ \frac{\theta+\varepsilon-t}{\varepsilon} & \theta \le t \le \theta + \varepsilon \\ \frac{t}{\varepsilon} & \theta + \varepsilon \le t \le 1 \end{cases}$$

It is piecewise continuous (on the obvious pieces) and they agree on boundaries, so it is continuous. One also sees that $\int_0^1 f(t)dt = \theta$. Its partner is defined as follows

$$g(t) = \begin{cases} \sqrt{f(t) - f(t)^2} & \theta \le t \le \theta + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

(2.1) is clear from $\varepsilon < \theta$; the supports of g and g_θ do not meet. By construction $f(t) + f(t-\theta) = 1$ for $t \in [\varepsilon, 2\theta]$. The support of g is contained in this interval, so (2.2) follows. $f(t) - f(t)^2$ is non-zero exactly on $(0, \varepsilon)$ and $(\theta, \theta + \varepsilon)$, which is also precisely when $g(t+\theta)$ and $g(t)$ are non-zero respectively, and this gives (2.3). Thus we have the projection $P = M_g V + M_f + M_{\overline{g}_{-\theta}}$ with $\tau(P) = \theta$. This is summarised in Figure 1.

In the case $\theta > \frac{1}{2}$, $1 - \theta < \frac{1}{2}$, so we can find find a projection P in $A_{1-\theta}$ with $\tau(P) = 1 - \theta$. But since $A_{1-\theta} \cong A_\theta$ and τ is unique, the corresponding P' in A_θ will be such that $\tau(P') = 1 - \theta$, so $1 - P'$ is our desired projection. Finally, for any α, with $[\alpha] = [k\theta]$, $C^*(U, V^k)$ is the irrational rotation algebra for $[k\theta] \in [0, 1]$. So we find P in it with $\tau(P) = \alpha$, but by uniqueness, this τ is simply the restricted τ from A_θ, and we are done. □

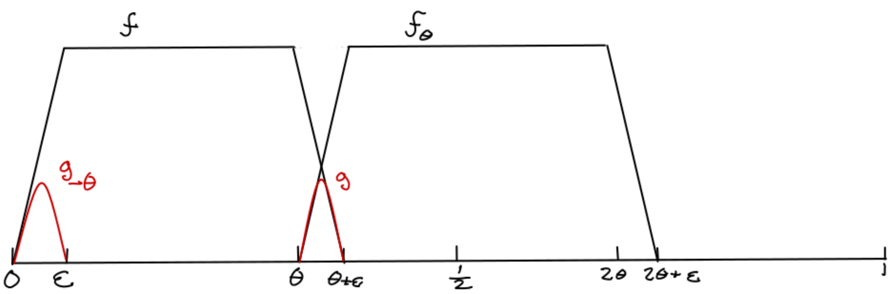

Figure 1: Graphs demonstrating the functional equations based on [1, Figure VI.1.1, p. 172]

Corollary 2.6. *The image of* $\tau_* : K_0(A_\theta) \to \mathbb{R}$ *contains* $\mathbb{Z} + \mathbb{Z}\theta$.

Proof. Any element $x \in \mathbb{Z} + \mathbb{Z}\theta$ can be written as $a + \alpha$, with $a \in \mathbb{Z}$ and α as in Theorem 2.5. The image of τ_* contains a since τ is a tracial state, and it contains α by the theorem, thus x is in the image. $\square$

3 The AF Algebra $\mathfrak{A}_\theta$

What remains for the proof of Theorem 1.3 is to show that the image of τ_* is contained in $\mathbb{Z} + \mathbb{Z}\theta$. This is accomplished by imbedding A_θ in an AF algebra constructed from the continued fraction expansion of θ. This idea is due to Pimsner and Voiculescu [3].

Let $\theta \in [0,1]$ an irrational number. Then there is a sequence $(a_n)_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty}[a_1, ..., a_n] = \theta$ where

$$[a_1, ..., a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{... + \frac{1}{a_n}}}}$$

There are corresponding sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$, $p_n \geq 0$ and $q_n > 0$ with $\frac{p_n}{q_n} = [a_1, .., a_n]$ for $n \geq 1$. Furthermore, we have $q_0 = 1$, $q_1 = a_1$, $p_0 = 0$, $p_1 = 1$ and the recursive relation

$$\begin{pmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{pmatrix} = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$$

for $n \geq 1$. It is not hard to see that $\frac{1}{q_n} \to 0$ as $n \to \infty$ geometrically fast. Since $\det\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = -1$, this implies that $\det\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = (-1)^{n-1}$ by induction. One consequence is that p_n and q_n are coprime. Breaking into even and odd cases, we obtain the inequalities

$$\frac{p_{2n-2}}{q_{2n-2}} < \frac{p_{2n}}{q_{2n}} < \frac{p_{2n+1}}{q_{2n+1}} < \frac{p_{2n-1}}{q_{2n-1}} \tag{3.1}$$

for all $n \geq 1$. This means that the even terms increase towards θ while the odd terms decrease towards θ.

Let for $n \geq 1$, let $X_n \subseteq X = \mathbb{R}/\mathbb{Z}$ be the subgroup generated by $\left[\frac{1}{n}\right]$. This is a discrete subspace containing n points, so $L^2(X_n)$ is generated by the n linearly independent elements $e^{2\pi kit}$, $0 \leq k < n$. Multiplication by $e^{2\pi it}$ shifts these elements around cyclically, which gives a unitary in $B(L^2(X_n)) \cong M_n(\mathbb{C})$ analogous to $U \in A_\theta$. Similarly, for some $0 \leq m < n$ $f(t) \mapsto f\left(t - \frac{m}{n}\right)$ gives a diagonal unitary analogous to V. In case $n = q_i$ and $m = p_i$ for some $i \geq 1$ we get U_n and V_n in $M_{q_n}(\mathbb{C})$ as above which will obey

$$U_n V_n = e^{2\pi i \frac{p_n}{q_n}} V_n U_n$$

Our goal will be to define $*$-homomorphisms $\varphi_n : M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C}) \to M_{q_{n+1}}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C})$ with multiplicites $\begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$ (notice this implies the map sends 1 to 1) such that $(U_n \oplus U_{n-1})_{n\geq 1}$ and $(V_n \oplus V_{n-1})_{n\geq 1}$ are Cauchy sequences in the inductive limit given by the φ_n, $\mathfrak{A}_\theta$. The first consequence of this is that the limits $U := \lim_{n\to\infty} U_n \oplus U_{n-1}$ and $V := \lim_{n\to\infty} V_n \oplus V_{n-1}$ will satisfy (1.1). Indeed, we have

$$UV = \lim_{n\to\infty} U_n V_n \oplus U_{n-1} V_{n-1} = \lim_{n\to\infty} e^{2\pi i \frac{p_n}{q_n}} V_n U_n \oplus e^{2\pi i \frac{p_{n-1}}{q_{n-1}}} V_{n-1} U_{n-1} = e^{2\pi i \theta} VU$$

U and V are also unitary, so by the universal property of A_θ, it imbeds in $\mathfrak{A}_\theta$. Since the multiplicites of the φ_n determine $\mathfrak{A}_\theta$, let us delay the construction of the φ_n to first demonstrate the consequences of this imbedding.

Proposition 3.1. *$K_0(\mathfrak{A}_\theta) = \mathbb{Z}^2$, $K_0(\mathfrak{A}_\theta)^+ \subseteq \{(x,y) \in \mathbb{Z}^2 : \theta x + y \geq 0\}$ and $(0,1)$ is the class of the identity map in $\mathfrak{A}_\theta$.*

Before we begin the computation, recall the following important fact.

Theorem 3.2. *Let $(A_i)_{i\geq 1}$ and $(\varphi_i)_{i\geq 1}$ be an inductive system of C^*-algebras with inductive limit $A, (\psi_i)_{i\geq 1}$, then $K_0(A), (\psi_{i*})_{i\geq 1}$ is the inductive limit of the induced inductive system of abelian groups $K_0(A_i))_{i\geq 1}, (\varphi_{i*})_{i\geq 1}$. Moreover, $K_0(A)^+ = \bigcup_{n\geq 1} \psi_{n*}\big(K_0(A_n)^+\big)$*

Proof. See [2, Theorem 6.3.2, p. 98] □

Proof of Proposition 3.1. Let $\mathfrak{A}_\mathfrak{n} := M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})$. We have that $\varphi_{n*} : K_0(\mathfrak{A}_n) = \mathbb{Z}^2 \to \mathbb{Z}^2 = K_0(\mathfrak{A}_{n+1})$ can be written with respect to the standard basis as the matrix $A_n = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$. These are all isomorphisms since the determinant is a unit in $\mathbb{Z}$, hence $K_0(\mathfrak{A}_\theta)$, the inductive limit by Theorem 3.2, is $\mathbb{Z}^2$ and all the canonical maps $\psi_{n*} : K_0(\mathfrak{A}_n) \to K_0(\mathfrak{A}_\theta)$ are isomorphisms. Since they are also positive maps, fix $\psi_{1*}(p_1, p_0)$ and $\psi_{1*}(q_1, q_0)$ as a basis of positive elements ($\begin{pmatrix} p_1 \\ p_0 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_0 \end{pmatrix}$ is a basis because the resulting column matrix has determinant ± 1. Thus the images by the isomorphism ψ_{1*} yield a basis). By the definition of $\mathfrak{A}_1$, this latter basis element is the image of the class of the identity map on $\mathfrak{A}_1$, so is the class of the identity on $\mathfrak{A}_\theta$. We then see that with respect to this basis,

$$\psi_{n*} = A_1^{-1} A_2^{-1} ... A_{n-1}^{-1} = \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}^{-1} = (-1)^{n-1} \begin{pmatrix} q_{n-1} & -q_n \\ -p_{n-1} & p_n \end{pmatrix}$$

Thus applying Theorem 3.2, every element of $K_0(\mathfrak{A}_\theta)^+$ is a positive linear combination of $(-q_{2n-1}, p_{2n-1})$ and $(q_{2n}, -p_{2n})$, $n \geq 1$. By the inequalities (3.1), $\theta(-q_{2n-1}) + p_{2n-1} \geq 0$ and $\theta q_{2n} - p_{2n} \geq 0$, so by (positive) linearity, $\theta x + y \geq 0$ for any $(x,y) \in K_0(\mathfrak{A}_\theta)^+$. □

Proposition 3.1 immediately gives us a state $\sigma : \big(K_0(\mathfrak{A}_\theta), K_0(\mathfrak{A}_\theta)^+, [1]\big) \to (\mathbb{R}, \mathbb{R}^+, 1)$ by $\sigma(x,y) = \theta x + y$ using our handy basis from the computation. Indeed, since $K_0(\mathfrak{A}_\theta)^+ \subseteq \{(x,y) \in \mathbb{Z}^2 : \theta x + y \geq 0\}$, it is a positive homomorphism, and $\sigma([1]) = \sigma(0,1) = 1$. This is very useful given the following theorem on AF algebras:

Theorem 3.3. *Let A be an AF algebra. Then the map sending a tracial state $\tau : A \to \mathbb{C}$ to a state $\tau_* : (K_0(A), K_0(A)^+, [1]) \to (\mathbb{R}, \mathbb{R}^+, 1)$ is a bijection.*

Proof. See [1, Theorem IV.5.3, p. 114]. □

The lift of σ to a trace will be the final piece of proof of Theorem 1.3. We must first, however, construct the desired $*$-homomorphisms φ_n. This will be done using a technique supplied by the following lemma:

Lemma 3.4. (Intertchange Lemma): *Let H be a Hilbert space and suppose elements e_i, f_i $1 \leq i \leq n$ form a orthonormal set, with $V = \operatorname{span}(e_0, .., e_{n-1}, f_0, ..., f_{n-1})$. Let $T \in B(H)$ with $Te_j = e_{j+1}$ and $Tf_j = f_{j+1}$ for $0 \leq j < n$. Then there exists an operator $S \in B(H)$ such $T = S$ on the orthoganal complement of V, $S\big(\operatorname{span}(e_j, f_j)\big) = \operatorname{span}\big(e_{j+1}, f_{j+1}\big)$ for $0 \leq j < n$, $S^n e_0 = f_n, S^n f_0 = e_n$ and $S^j e_0, S^j f_0$ form an orthonormal basis for V. Moreover $\|S - T\| < \frac{\pi}{n}$.*

Proof. We define S to simply be T on the orthogonal complement of V. The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is unitarily diaganolisable with eigenvalues $1, -1$. Thus it has an n-th root which is unitarily diagonalisable with eigenvalues $1, e^{\pi \frac{i}{n}}$. Let us denote is Θ. On the restriction $T_i : \operatorname{span}(e_i, f_i) \to \operatorname{span}(e_{i+1}, f_{i+1})$ is the identity matrix with repsect to the obvious bases. Thus we define the analogous S_i as Θ. This defines S on all of H.

By definition S agrees with T on the orthogonal complement of V and it sends $\operatorname{span}(e_j, f_j)$ onto $\operatorname{span}(e_{j+1}, f_{j+1})$ for $0 \leq j < n$. Since $\Theta^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $S^n e_0 = f_n$ and $S^n f_0 = e_n$. Since the $\operatorname{span}(e_i, f_i)$ are all orthogonal to one another and Θ is unitary, the $S^i e_0, S^i f_0$ are clearly an orthonormal set, and by dimension must be a basis for V. Finally,

$$\|S - T\| = \|\Theta - I_2\| = |e^{\pi \frac{i}{n}} - 1| < \frac{\pi}{n}$$

□

Notice that applying Lemma 3.4 multiple times on an operator T for mutually orthogonal orthonormal sets will result in an operator S with $\|T - S\| < \frac{\pi}{n}$, where n is cardinality of the smallest orthonormal set used.

Proposition 3.5. *There is an element $U'_{n+1} \in M_{q_{n+1}}(\mathbb{C})$ which is unitarily conjugate to $U_n^{(a_{n+1})} \oplus U_{n-1}$ with $\|U_n - U'_n\| < \frac{2\pi}{q_{n-1}}$.*

Proof. Let e_j be the elements of $\mathbb{C}^{q_{n+1}} = B\big(L^2\big(X_{q_{n+1}}\big)\big)$ given by the $e^{2\pi i j t}$. Hence, $e_j = e_{j+q_{n+1}}$ and $U_{n+1} e_j = e_{j+1}$. Let $b = \lfloor \frac{q_n}{2} \rfloor$, $b' = \lceil \frac{q_n}{2} \rceil = q_n - b$ and $s = \lceil \frac{q_{n-1}}{2} \rceil$. For each $1 \leq k \leq a_{n+1}$ we use the Interchange Lemma on U_{n+1} and $f_j = e_{-kb+j}$, $g_j = e_{kb'+j}$, $0 \leq j \leq s$, to define U'_{n+1}. Clearly, we have $\|U_n - U'_n\| < 2\frac{\pi}{q_{n-1}}$. It is now enough to exhibit a_{n+1} cycles of U'_{n+1} of size q_n and one of size q_{n-1} which together form an orthonormal basis of $\mathbb{C}^{q_{n+1}}$.

First start with $g_j^1 = e_{-b+j}$ for $s \leq j \leq q_n$. U'_{n+1} sends g_j^1 to g_{j+1}^1 for $s \leq j < q_n$. $g_{q_n}^1 = e_{b'}$, so for $0 \leq j < s$, let $g_j^1 \in \operatorname{span}(e_{-b+j}, e_{b'+j})$ be the elements obtained by applying U'_{n+1} starting with $g_0^1 := g_{q_n}^1 = e_{b'}$. Then by the interchange, $U'_{n+1} g_{s-1}^1 = g_s^1$, so we have a cycle of length q_n and the elements form on orthonormal set. Now for $1 < k \leq a_{n+1}$, we start with $g_j^k = e_{-kb+j}$ for $s \leq j \leq b$. Then, from the $(k-1)$-th interchange, U'_{n+1} shifts vectors $g_j^k \in \operatorname{span}\big(e_{-(k-1)b+(j-b)}, e_{(k-1)b'+(j-b)}\big)$ for $b \leq j < b + s$, starting at $g_b^k = e_{-(k-1)b}$ and ending at $g_{b+s}^k = e_{(k-1)b'+s}$. Then U'_{n+1} takes $g_j^k := e_{(k-1)b'+(j-b)}$ to g_{j+1}^k for $b + s \leq j \leq b + b' = q_n$. So $g_{q_n}^k = e_{kb'}$, and so the k-th interchanges means U'_{n+1} takes elements $g_j^k \in \operatorname{span}(e_{-kb+j}, e_{kb'+j})$ to g_{j+1}^k, $0 \leq j < s$, starting with $g_0^k := g_{q_n}^k = e_{kb'}$ and ending up at $g_s^k = e_{-kb+s}$, so we get a cycle of length q_n. Despite using the $(k-1)$-th interchange, this cycle is still orthogonal to the $k-1$-th cycle since they start on "opposite sides" of it. Finally, let $g_j^0 = e_{a_{n+1}b'+j}$ for $s \leq j \leq q_{n-1}$. Then $U'_{n+1} g_j^0 = g_{j+1}^0$ for $s \leq j < q_n - 1$. Since $q_{n+1} = a_{n+1} q_n + q_{n-1}$, $a_{n+1} b' + q_{n-1} \equiv -a_{n+1} b \bmod q_{n+1}$, so the a_{n+1}-th interchange means there are elements $g_j^0 \in \operatorname{span}\big(e_{-a_{n+1}b+j}, e_{a_{n+1}b'+j}\big)$ which U'_{n+1} shifts to g_{j+1}^0, $0 \leq j < s$, starting with $g_0^0 := g^0_{a_{n+1}b'+q_{n-1}=e_{-a_{n+1}b}}$ and ending at $e_{a_{n+1}b'+s}$, giving a cycle of length q_{n-1}. □

Keeping the notation from the proof of Proposition 3.5, for $1 \leq k \leq a_{n+1}$, g_j^k makes sense for any $j \in \mathbb{Z}$ by extending using q_n-periodicity, and U'_{n+1} shifts by $+1$ on j. The same holds for g_j^0 but with q_{n-1}-periodicity. So letting e_j^k $(1 \leq k \leq a_{n+1}, 0 \leq j < q_n)$ and e_j^0 $(0 \leq j < q_{n-1})$ denote the basis for a block diagonal decomposition of a $q_{n+1} \times q_{n+1}$ matrix into a_{n+1} $q_n \times q_n$ blocks and one $q_{n-1} \times q_{n-1}$ block, We can write U'_{n+1} as $W\big(U_n^{(a_n+1)} \oplus U_{n-1}\big)W^*$ with the unitary $W \in M_{q_{n+1}}(\mathbb{C})$ sending e_j^k to $g_{j+c_k}^k$, for any fixed integers c_k, $0 \leq k \leq a_{n+1}$.

Given a choices of c_k, we can define

$$\varphi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}, \qquad B_n \oplus B_{n-1} \mapsto W\big(B_n^{(a_{n+1})} \oplus B_{n-1}\big)W^* \oplus B_n$$

Since W is unitary this is a $*$-homomorphism with multiplicites $\begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$.

Proposition 3.6. *There is a choice of c_k such that $\|V_{n+1} - \varphi_n(V_n \oplus V_{n-1})\| < \frac{4\pi}{q_n} + \frac{2\pi}{q_{n-1}}$.*

Proof. For simplicity, let $\lambda_k = e^{-2\pi i \frac{p_k}{q_k}}$ for $k \geq 1$. Let b, b' and s be as in the proof of Proposition 3.5. p_{n-1} and q_{n-1} are coprimes, so λ_{n-1} generates the cyclic subgroup of order q_{n-1} in $\mathbb{T}$, thus we can find a $m \in \mathbb{Z}$ such that $|\lambda_{n-1}^m - \lambda_{n+1}^{a_{n+1}b'}| < \frac{2\pi}{q_{n-1}}$. Set $c_0 = m$. We set $c_k = -kb$ for $k \geq 1$. This will mean that $V'_{n+1} g_j^k = \lambda_n^{j-kb} g_j^k$.

Let $V'_{n+1} = \varphi_n(V_n \oplus V_{n-1})$. Then we see from the proof of Proposition 3.5 that V'_{n+1} has $g_j^k = e_{-kb+j}$, $s \leq j \leq b$ and $g_{j+b}^k = e_{(k-1)b'+j}$, $s \leq j \leq b'$, for any $1 \leq k \leq a_{n+1}$. There is also $g_j^0 = e_{a_{n+1}b'+j}$ for $s \leq j \leq q_{n-1}$. These are all also eigenvectors of V_{n+1} by definition. Moreover, the subspaces $E_{k,j} = \operatorname{span}\big(e_{-kb+j}, e_{kb'+j}\big) = \operatorname{span}(g_j^k, g_{j+b}^{k+1}$ for $0 < j < s$ and $1 \leq k < a_{n+1}$ are stable for both V'_{n+1} and V_n. Indeed, the elements of the sequences g^k and g^{k+1} forming these subspaces are precisely the elements spanning each step of the k-th interchange, with one sequence going from the f_i to g_i (as in the notation for the interchanges from the proof of Proposition 3.5), and the other going the opposite direction. Finally, we have the subspaces $E_{0,j} = \operatorname{span}\Big(e_{-a_{n+1}b+j}, e_{a_{n+1}b'+j}\Big) = \operatorname{span}\Big(g_j^{a_{n+1}}, g_j^0\Big)$ for $0 < j < s$, which are also fixed by both V_{n_1} and V'_{n+1} by the same argument. The $E_{k,j}$ and 1-dimensional subspaces generated by the above eigenvalues are mutually orthogonal and stable under $V_{n+1} - V'_{n+1}$, so we can proceed by estimating the error on each one individually.

We start with the cases where $k \geq 1$. For $e_{-kb+j} = g_j^k$, $s \leq j \leq b$, due to our choice of c_k, we have $\|(V'_{n+1} - V_{n+1})e_{-kb+j}\| = |\lambda_n^{j-kb} - \lambda_{n+1}^{j-kb}|$. In general, $|e^{2\pi i \frac{a}{m}} - e^{2\pi i \frac{a'}{m}}| < \frac{2\pi}{m}\ |a - a'|$. So putting $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ on a common denominator, we get $|\lambda_n^{j-kb} - \lambda_{n+1}^{j-kb}| < \frac{2\pi}{q_n q_{n+1}}\ |j - kb|$, since $p_n q_{n+1} - q_n p_{n+1} = \pm 1$. $|j - kb| < |a_{n+1} b|$, so we get an upper bound of $\frac{2\pi}{q_n}$. Similarly, for $g_{j+b}^k = e_{(k-1)b'+j}$, $s \leq j \leq b'$, we also have $\|(V'_{n+1} - V_{n+1})e_{kb'+j}\| < \frac{2\pi}{q_n}$. On $E_{k,j}$, $0 < j < s$, we have

$$V'_{n+1} g_j^k = \lambda_n^{j-kb} g_j^k \qquad V'_{n+1} g_{j+b}^{k+1} = \lambda_n^{j+b-(k+1)b} g_{j+b}^{k+1} = \lambda_n^{j-kb} g_{j+b}^{k+1}$$

So V'_{n+1} just acts as $\lambda_n^{j-kb} I_2$ on $E_{k,j}$. Furthermore, $kb' \equiv -kb \bmod q_n$, so $\lambda_n^{kb'+j} = \lambda_n^{j-kb}$. Hence $\|(V'_{n+1} - V_{n+1})e_{kb'+j}\| = |\lambda_n^{kb'+j} - \lambda_{n+1}^{kb'+j}| < \frac{2\pi}{q_n}$ and likewise for $\|(V'_{n+1} - V_{n+1})e_{-kb+j}\|$, which gives the same bound for the $E_{k,j}$.

To finish we consider the case $k = 0$. On $e_{a_{n+1}b'+j} = g_j^0$, $s \leq j \leq q_{n-1}$, we have

$$\|(V'_{n+1} - V_{n+1})e_{a_{n+1}b'+j}\| = |\lambda_{n-1}^{j+c_0} - \lambda_{n+1}^{a_{n+1}b'+j}| \leq |\lambda_{n-1}^{c_0} - \lambda_{n+1}^{a_{n+1}b'}\ | + |\lambda_{n-1}^{j} - \lambda_{n+1}^{j}|$$

The first term is bounded by $\frac{2\pi}{q_{n-1}}$. For the second one, we must must estimate $\frac{2\pi}{q_{n-1}q_{n+1}} j\ |p_{n-1}q_{n+1} - q_{n-1}p_{n+1}|$. $j \leq q_{n-1}$ and

$$p_{n-1}q_{n+1} - q_{n-1}p_{n+1} = a_{n+1}p_{n-1}q_n + p_{n-1}q_{n-1} - a_{n+1}q_{n-1}p_n - q_{n-1}p_{n-1} = \pm a_{n+1}$$

so $\frac{2\pi}{q_{n-1}q_{n+1}} j\ |p_{n-1}q_{n+1} - q_{n-1}p_{n+1}| \leq \frac{2\pi a_{n+1}}{q_{n+1}} \leq \frac{2\pi}{q_n}$, for a total bound of $\frac{2\pi}{q_{n-1}} + \frac{2\pi}{q_n}$. On $E_{0,j}$,

$$\|V_{n+1}|E_{0,j} - \lambda_{n+1}^{j-a_{n+1}b} I_2\| = |\lambda_{n+1}^{a_{n+1}b'+j} - \lambda_{n+1}^{j-a_{n+1}b}| = |\lambda_{n+1}^{a_{n+1}q_n} - 1| < \frac{2\pi}{q_n}$$

and

$$\|V'_{n+1}|E_{0,j} - \lambda_{n+1}^{j-a_{n+1}b} I_2)\| = \max\Big(|\lambda_n^{j-a_{n+1}b} - \lambda_{n+1}^{j-a_{n+1}b}|, |\lambda_{n-1}^{j+c_0} - \lambda_{n+1}^{j-a_{n+1}b}|\Big)$$

This is the maximum of $\frac{2\pi}{q_n}$ and $\frac{2\pi}{q_{n-1}} + \frac{2\pi}{q_n}$, which is the latter. So by the triangle inequality, the error on $E_{0,j}$ is bounded by $\frac{2\pi}{q_{n-1}} + \frac{4\pi}{q_n}$. This is the largest of the bounds obtained so far, so we conclude that $\|V'_{n+1} - V_n\| < \frac{2\pi}{q_{n-1}} + \frac{4\pi}{q_n}$, as desired. □

Corollary 3.7. *The classes of $U_n \oplus U_{n-1}$ and $V_n \oplus V_{n-1}$ in $\mathfrak{A}_\theta$ form Cauchy sequences in $\mathfrak{A}_\theta$.*

Proof.

$$\|U_n \oplus U_{n-1} - U_{n+1} \oplus U_n\| \leq \|\varphi_n(U_n \oplus U_{n-1}) - U_{n+1} \oplus U_n\| \leq \|U'_{n+1} - U_{n+1}\| \leq \frac{2\pi}{q_{n-1}}$$

and similarly we see that $\|V_n \oplus V_{n-1} - V_{n+1} \oplus V_n\| < \frac{2\pi}{q_{n-1}} + \frac{4\pi}{q_n}$. Both of these bounds decrease geometrically fast so it follows the sequences are Cauchy. □

Proof of Theorem 1.3. As discussed earlier, Corollary 3.7 implies that there is an imbedding $i : A_\theta \to \mathfrak{A}_\theta$. By Theorem 3.3, there is a tracial state $\mathfrak{t} : \mathfrak{A}_\theta \to \mathbb{C}$ with $\mathfrak{t}'_* = \sigma$, the state from Proposition 3.1. Then by uniqueness, $\tau = \mathfrak{t} \circ i$. It is clear from the definitions of the induced maps on K_0 that $\tau_* = \sigma \circ i_*$, and so it follows that the image of τ_* is contained in $\mathbb{Z} + \mathbb{Z}\theta$, hence equals $\mathbb{Z} + \mathbb{Z}\theta$ by Theorem 2.5. This gives $(i) \Rightarrow (iii)$, and so proves our theorem. □

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